Math 3210 Tutorial 5

General Steps on Simplex methods:

(1) Find a starting BFS

$\downarrow$

(2) Determined entering variables (make sure that the new solution is better)

$\downarrow$

(3) Determine leaving variable (make sure that the new corner point is feasible)

$\downarrow$

(4) move from one BFS $\to$ next BFS

General Directions Example 1: Try to find a basic solution for the following system

$$\text{Max } 4X_1 - X_2 = f(x_1, x_2) = f(x)$$

Subject to

$$X_1 \le 4$$

$$X_2 \le 6$$

$$x_1 + X_2 \le 8$$

$$x_1 \ge 0 \quad x_2 \ge 0$$

$$X_1 + s_1 \qquad \le 4$$

$$X_2 + \qquad s_2 \qquad \le 6$$

$$x_1 + x_2 + \qquad s_3 \le 8$$

$$C = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$f(x) = f(\vec{x}_0) + \vec{c}\,(\vec{x} - \vec{x}_0)$

regard as position vectors.

chose the part that

$\vec{c}$

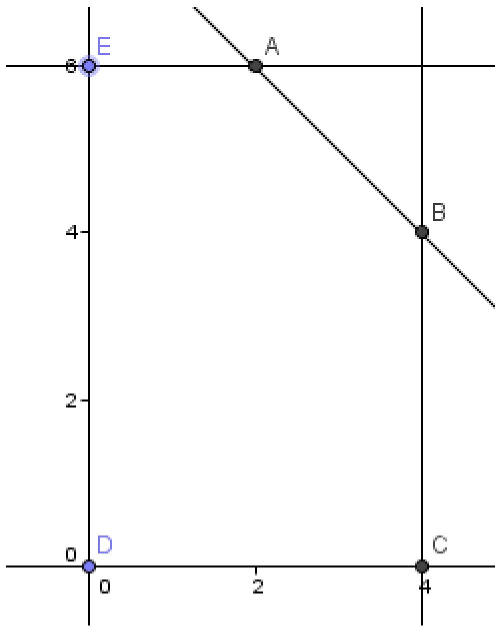
When we go from one point to the next.

$x \to x_0$

~~We may~~ $f(x) = f(x_0 + x - x_0)$

$= f(x_0) + f(x - x_0)$

$$f(x) = \underbrace{f(x_0)}_{\text{old solution}} + c \cdot (x - x_0)$$



When we go from one basic solution to the other

let innitially we have

$$\begin{pmatrix} x_{B_1} \\ \vdots \\ x_{B_m} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} B & \vdots & \end{pmatrix}}_{A} \begin{pmatrix} x_{B_1} \\ \vdots \\ x_{B_m} \\ \vdots \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

dummy

$\cancel{f(x) = C_{B_1} x_{B_1}}$ $\quad f(x) = C_{B_1} x_{B_1} \cdots C_{B_m} x_{B_m}$

$$y = B^{-1} A$$

$$= \begin{pmatrix} I & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} y_{B_1} & y_{B_2} & \cdots & y_{B_m} & y_{B_{m+1}} & \cdots & y_{B_{m+n}} \end{pmatrix}$$

Back to example:

$$C = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \rightarrow C = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$b = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

We have one basic solution

$$x_1 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

do some re arragement.

$$X_1^{B'} = \begin{pmatrix} 4 \\ 6 \\ 8 \\ 0 \\ 0 \end{pmatrix} \qquad c^{B'} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ -1 \end{pmatrix}$$

matching
$b_1 = x_3$
$b_2 = x_4$
$b_3 = x_5$
$b_4 = x_1$
$b_5 = x_2$

$$A^{B'} = \begin{pmatrix} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{B} & \begin{matrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{matrix} \end{pmatrix}$$

leave space for c_{B_1}

$$B^{-1} = B = I$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

If X_{B_r} is leaving, X_{B_j} is entering

$$C^T(\vec{X} - \vec{X}_0) = \frac{X_{B_r}}{y_{rj}}\{C_j - Z_j\}$$

new comer

$\boxed{\text{where } Z_j = C_B \cdot y_{B_j}}$

need to make sure that $(C_j - Z_j)$ is maximised

~~Poter~~ Potential variables to enter.

$X_{B'_4}$ $\qquad$ X_{B_5}

$Z_{B'_4} = C^{B'} \cdot y_{B_4}$ $\qquad$ $Z_{B'_5} = C^{B'} \cdot y_{B_5}$

$= 0$ $\qquad$ $= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$C_{B'_4} - Z_{b_4}$ $\qquad$ $C_{B'_5} - Z_{b_5}$

$= 4$ $\qquad$ $= -1$

X_{B_4} is entering. $\rightarrow$ X_1 is entering

Check which variable is leaving (feasibility ·)

~~m~~ target: minimized $\dfrac{x_{B_r}}{\underbrace{y_{rj}}_{\text{the } j \text{ is determined}}}$

$\cancel{x_{B}}$

$j = B_4$

$$y_{B_4} = \begin{pmatrix} \boxed{1} \\ 0 \\ 1 \end{pmatrix} \quad x_{B_r} = \begin{pmatrix} \boxed{4} \\ 6 \\ 8 \\ 0 \\ 0 \end{pmatrix}$$

~~the~~ j i

$$\frac{x_{B_1}}{y_{1,B_4}} = 4 \qquad \frac{x_{B_2}}{y_{2,B_4}} = \infty$$

$$\frac{x_{B_3}}{y_{1,B_4}} = 8.$$

x_{B_1} is leaving.

Final step: Move it from one basic solution to the other

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix}$$

Method 1:

Consider

~~A~~ $AX = b$

$$\begin{array}{c} \text{除} \\ \begin{pmatrix} 1 & \boxed{0 \;\; 1} & 0 & 0 \\ 0 & 1 \;\; 0 & 0 & 0 \\ 1 & 1 \;\; 0 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ 0 \\ 0 \\ \cancel{x_4} \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

Another Example on 3 by 3 system

Example 2:

Ex 2) Max $5x_1 - 2x_2 + 3x_3 = f(x_1, x_2, x_4)$

subject to

$$x_1 + 3x_2 + 3x_3 \le 11$$
$$x_1 + 4x_2 + 3x_3 \le 9$$
$$x_1 + 3x_2 + 4x_3 \le$$

$$x_1, x_2, x_4 \le 0.$$

$$\begin{pmatrix} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \\ 13 \end{pmatrix}$$

start with the origin. (s_1, s_2, s_3) as basic

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 & 3 & 3 \\ 0 & 1 & 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 1 & 3 & 4 \end{pmatrix}}_{B} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \\ 13 \end{pmatrix}$$

$$C = \begin{pmatrix} 5 \\ -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ -2 \\ 3 \end{pmatrix} \qquad X_{B_1} = \begin{pmatrix} 11 \\ 9 \\ 13 \end{pmatrix}$$

$X_{B_1'} = s_1$
$X_{B_2'} = s_2$
$X_{B_3'} = s_3$
$X_{B_4'} = x_1$
$X_{B_5'} = x_2$
$X_{B_6'} = x_3$

Got a Basic solution: Then try seeing what new variable to enter

$$\cancel{C_B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ -2 \\ 3 \end{pmatrix}} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad C_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Y = B^{-1}A = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 3 \\ 0 & 1 & 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 1 & 3 & 4 \end{array}\right) \qquad X_{B'} = \begin{pmatrix} 11 \\ 9 \\ 13 \end{pmatrix}$$

Potential variables $X_{B_4}, X_{B_5}, X_{B_6}$

$$Z_{B_4} = Z_{B_5} = Z_{B_6} = 0 \qquad C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ -2 \\ 3 \end{pmatrix}$$

$$\boxed{C_{B_4} - Z_{B_4} = 5}$$

$C_{B_5} - Z_{B_5} = 0 - 2$

$C_{B_6} - Z_{B_6} = 3$

determine which is leaving

$$y_{B_4} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad X_{B_r} = \begin{pmatrix} 11 \\ 9 \\ 13 \end{pmatrix} \qquad \text{minimise } \frac{X_{B_r}}{y_{B_r,4}} \Rightarrow \text{chose } X_{B_2} \text{ to leave.}$$

(Easier) Way.

$$11 A_{B_1} + 9 A_{B_2} + 13 A_{B_3} = \begin{pmatrix} 11 \\ 9 \\ 13 \end{pmatrix} \quad \text{— (2)}$$

need to replace A_{B_2} with A_{B_4}.

$$A_{B_1}, A_{B_2}, A_{B_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_{B_4} = A_{B_1} + A_{B_2} + A_{B_3}$$

$$A_{B_2} = A_{B_4} - A_{B_1} - A_{B_3} \quad \text{— (1)}$$

Sub (1) into (2).

$$2 A_{B_1} + 9 A_{B_4} + 4 A_{B_3} = \begin{pmatrix} 11 \\ 9 \\ 13 \end{pmatrix}$$

$$A^{B^2} = \begin{pmatrix} 1 & 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 & 4 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 \\ 5 \\ 0 \\ -2 \\ 3 \\ 0 \end{pmatrix}$$

$$X_{B^2} = \begin{pmatrix} s_1 \\ x_1 \\ s_3 \\ x_2 \\ x_3 \\ s_2 \end{pmatrix} = \begin{pmatrix} x_{B_1^2} \\ x_{B_2^2} \\ x_{B_3^2} \\ x_{B_4^2} \\ x_{B_5^2} \\ x_{B_6^2} \end{pmatrix}$$

keep record. $C_{B^2} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

$$B^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad B^{2^{-1}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad Y^2 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 4 & 3 & 1 \\ 0 & 0 & 1 & -4 & -1 & -1 \end{array}\right)$$

$$Z_{B^2_4} = C_{B^2} \cdot Y^2_{B_4} = 20 \qquad C_{B_4} - Z_{B_4} = -22$$
$$Z_{B^2_5} = \quad - \quad = 19 \qquad C_{B_5} - Z_{B_5} = -12$$
$$Z_{B^2_6} = \quad -- \quad = 5 \qquad C_{B_6} - Z_{B_6} = -5$$

We can also see that the solution we get.

$x_1 = 5 \quad x_2 = x_3 = 0$ ~~$= x_3 = 0$ is optimal point~~

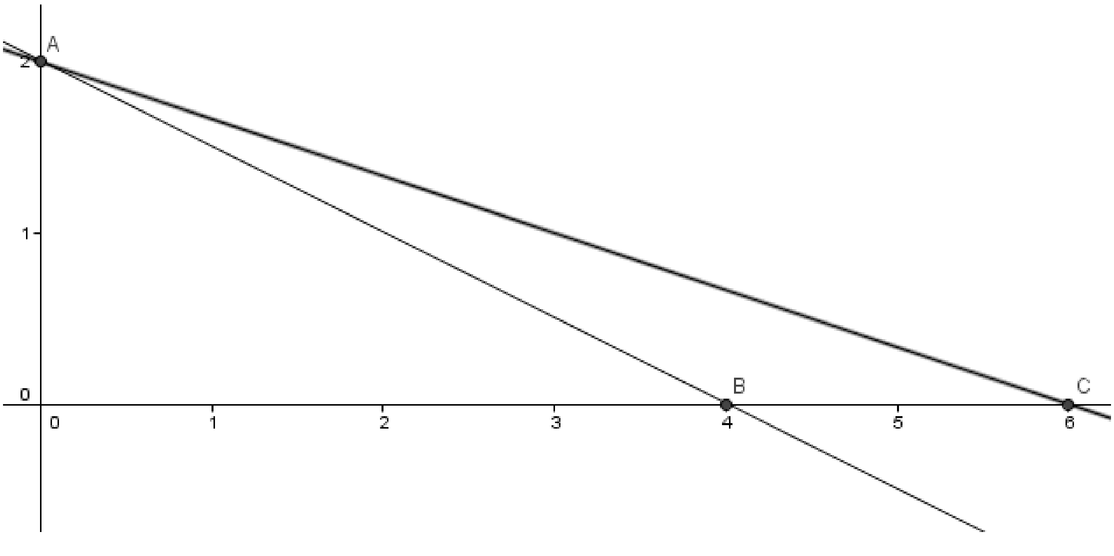
is optimal.

Example 3: what happens when degenerating solution occurs?

$$A = \begin{pmatrix} 2 & 4 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 8 \\ 6 \end{pmatrix} \quad C = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad C_B = \begin{pmatrix} 5 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad C_B = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$



$$B = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1.5 & -2 \\ -0.5 & 1 \end{pmatrix}$$

$$\begin{array}{c} Y = \begin{pmatrix} 1 & 0 & 1.5 & -2 \\ 0 & 1 & -0.5 & 1 \end{pmatrix} \\ \qquad\qquad\quad Y_{s_1} \quad Y_{s_2} \end{array}$$

$$f(x) = f(x_0) + c(x - x_0)$$
$$= f(x_0) + \underbrace{\frac{x_{Br}}{y_{rj}}}_{0}\underbrace{(x - x_0)}_{0}$$

$Z_{s_1} = 9.5 \qquad c_{s_1} - 9.5 = -9.5$

$Z_{s_2} = -3.5 \qquad c_{s_2} - (-3.5) = 3.5$

Put in S_2, take out x_1

We chose which variable to kick out by choosing $\min\left(\frac{x_{Br}}{y_{rj}}\right)$

$$\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} s_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

$s_1 = 0$ as well